HOMOGENEOUS LEAST SQUARES PROBLEM

Keijo Inkilä
Helsinki University of Technology
Laboratory of Photogrammetry and Remote Sensing
P.O.Box 1200, FIN-02015 TKK
keijo.inkila@tkk.fi

ABSTRACT

The homogeneous Least Squares Problem is first defined and discussed. Then two methods for solving the LS-problem are presented. These methods are based on the use of the generalized eigenvalue decomposition and the generalized singular value decomposition, respectively. Finally, a numerical example is given.

1. INTRODUCTION

The solution of a (non-homogeneous) linear system \( Ax = b \) is a typical problem in photogrammetry. Usually, the system is, in addition, overdetermined (i.e. \( Ax \approx b \)) and the existence of the solution is ensured by the Least Squares condition. The solution of a homogeneous linear system \( Ax = 0 \) is a less common problem. However, some interesting applications exist, where homogeneous systems naturally arise.

First, the equation \( ax + by + c = 0 \) represents a straight line in xy-plane. If the points \((x_i, y_i), i = 1, ..., n\), lie (exactly) on the line, we have

\[
\begin{bmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  \vdots & \vdots & \vdots \\
  x_n & y_n & 1 \\
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
\end{bmatrix}
\]

i.e. a homogeneous system \( Ax = 0 \) with \( A \) a \( n \times 3 \)-matrix. Similarly, making the substitutions into the equation \( ax^2 + by^2 + cxy + dx + ey + f = 0 \) of a conic section (e.g. circle or ellipse) leads also to a homogeneous system.

In photogrammetry and computer vision there are two important examples, where solution of a homogeneous system is needed (see e.g. (Mikhail et al., 2001) and (Forsyth et al., 2003)):

- The determination of the parameters of the Direct Linear Transformation (DLT).
- The determination of the essential or fundamental matrix from the given (measured) coordinates of the corresponding points in two images.
2. THE HOMOGENEOUS LS-PROBLEM

The solution of any homogeneous system $Ax = 0$ has the following special features:

- There is always the trivial solution $x = 0$, but it is not interesting.
- If $x \neq 0$ is a solution, then $kx$, where $k$ is an arbitrary scalar, is also a solution (only the ratios of the unknowns can be determined).

An additional condition is thus required to fix the length of the solution vector (to exclude the trivial solution). A simple way is to fix some component of the solution vector, say $x_j = 1$, and to solve the rest of the unknowns $\bar{x}$ from the non-homogeneous system $A\bar{x} = -A_j$. This may, however, lead to difficulties: If the (true) value of $x_j$ happens to be zero, there exists no scalar $k \neq 0$ such that $kx_j = 0$. It is thus better to use a more general condition

$$\|Bx\| = 1$$

where $B$ is a $p \times u$-matrix (for simplicity, we choose $\|Bx\| = 1$ instead of $\|Bx\| = \alpha \neq 1$, because the scaling is not critical). Note that the condition $\|Bx\| = \sqrt{x^T B^T B x} = 1$ contains the following special cases:

- If $B = I$, then $\|Bx\| = \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_u^2} = 1$
- If $B = [I_p \ 0]$, then $\|Bx\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2} = 1$
- If $B = \text{diag}(b_1,b_2,\ldots,b_u)$, then $\|Bx\| = \sqrt{b_1^2 x_1^2 + b_2^2 x_2^2 + \cdots + b_u^2 x_u^2} = 1$

We turn now to the second difficulty. It is obvious, that the homogeneous system $Ax = 0$ has no solution (other than $x = 0$), if $n \geq u$ and the rank of $A$ is full, i.e. $\text{rank}(A) = u$ (typical case in practical problems!). It is then natural to search for a solution that makes the contradiction “small”, i.e. $Ax \approx 0$. A popular condition is the Least Squares criterion

$$\|Ax\|^2 = \min$$

Based on the preceding considerations we state now the following problem:

**Given a $n \times u$-matrix $A$:**

$$n \geq u \quad \text{and} \quad \text{rank}(A) = u \quad (1)$$

**find the LS-solution $\hat{x}$, that fulfills the conditions:**

$$\|Ax\|^2 = \min \quad (2)$$

and

$$\|B\hat{x}\| = 1 \quad (3)$$

This problem here is called the homogeneous least squares problem.
3. THE SOLUTION OF THE HOMOGENEOUS LS-PROBLEM

In this chapter we present two solution methods for the homogeneous LS-problem. The core of these methods is the computation of

- the generalized eigenvalue decomposition or
- the generalized singular value decomposition.

These decompositions are briefly prescribed in the Appendix. Mathematical details can be found e.g. in (Golub et al., 1991). It is important from practical point of view that many mathematical software packages such as MATLAB contain these decompositions (Mathworks, 2005).

3.1 Solution using the generalized eigenvalue decomposition

The solution that fulfills the conditions $\|Ax\| = \min$ and $\|Bx\| = 1$ can be found using the Lagrange method, i.e. by minimizing

$$L(x, \lambda) = \|Ax\|^2 + \lambda (1 - \|Bx\|^2)$$

where $\lambda$ is the Lagrange multiplier. Setting the partial derivatives of $L$ with respect to $x$ and $\lambda$ to zero gives the equations

$$A^T Ax = \lambda B^T Bx$$
$$\|Bx\|^2 = 1$$

It can be seen immediately that the first condition (5) is satisfied by the generalized eigenvectors of the symmetric matrices $A^T A$ and $B^T B$. Using $A^T Ax = \lambda B^T Bx$ we further have

$$L = \|Ax\|^2 + \lambda (1 - \|Bx\|^2) = x^T A^T Ax + \lambda (1 - x^T B^T Bx) = \lambda$$

Thus, the solution that minimizes $L$ is, in particular, the generalized eigenvector $e_{\min}$ related to the smallest generalized eigenvalue $\lambda_{\min}$, i.e. $\hat{x} = e_{\min}$. In general, $\|Be_{\min}\|^2 = \alpha \neq 1$, therefore, in order to fulfill the condition $\|B\hat{x}\| = 1$, a simple scaling is required:

$$\hat{x} = (1/\|Be_{\min}\|) e_{\min}$$
In summary, we have the following solution algorithm:

1. Compute the matrices $M = A^T A$ and $N = B^T B$
2. Compute the generalized eigenvalue decomposition $M E = N E D$ of $M$ and $N$.
3. Search for the smallest generalized eigenvalue $\lambda_{\text{min}}$ and choose the related eigenvector $e_{\text{min}}$ as the LS-solution, that is $\hat{x} = e_{\text{min}}$.
4. Make an appropriate scaling, if necessary (e.g. $\hat{x}_{\text{scaled}} = (1/\|B e_{\text{min}}\|) e_{\text{min}}$).

There are two important special cases, where it is sufficient to compute the (ordinary) eigenvalue decomposition instead of the generalized eigenvalue decomposition. First, if $B = I$ (find $\hat{x}$, such that $\|A\hat{x}\| = \text{min}$ and $\|\hat{x}\| = 1$) the condition (5) reduces to the more simple condition

$$A^T A \hat{x} = \lambda \hat{x}$$

(9)

The eigenvectors of $A^T A$ satisfy this equation. But, using $A^T A x = \lambda x$

$$\|A x\|^2 + \lambda (1 - \|x\|^2) = x^T A^T A x + \lambda (1 - x^T x) = x^T (A^T A x - \lambda x) + \lambda = \lambda$$

Thus, the LS-solution is the eigenvector corresponding to the smallest eigenvalue of $A^T A$. An alternative derivation for this result is given in (Forsyth et al., 2003).

The second interesting special case is $B = [0 \ I_p]$, i.e. the constraint contains only a part of the parameters (the last $p$ parameters here). By partitioning $A = [A_1 \ A_2]$ and $x = [x_1^T \ x_2^T]^T$ we have the conditions:

$$\|A x\| = \|A_1 x_1 + A_2 x_2\| = \text{min} \quad \text{and} \quad \|x_2\| = 1$$

(10)

The LS-solution can be found in this case by performing the following steps:

1. Compute the coefficient matrix of the reduced normal equation ($x_1$ eliminated!)

$$N_2 = A_1^T A_2 - (A_1^T A_2)^T (A_1^T A_1)^{-1} (A_1^T A_2)$$

(11)

2. Compute the eigenvalue decomposition of $N_2$.

3. Choose the eigenvector corresponding to the smallest eigenvalue as the LS-solution, i.e.

$$\hat{x}_2 = e_{\text{min}}$$

(12)

4. Solve $\hat{x}_1$ from the equation

$$A_1^T A_1 \hat{x}_1 = -A_1^T A_2 \hat{x}_2$$

(13)
3.2 Solution using the generalized singular value decomposition

Let us assume that matrices $A$ and $B$ have the generalized singular value decomposition

$$A = U S_A F^T$$
$$B = V S_B F^T$$

where $U$ is an orthogonal nxn-matrix ($U^T U = I$), $V$ is an orthogonal pxp-matrix ($V^T V = I$), $S_A$ is a “diagonal” nxu-matrix, $S_B$ is a “diagonal” pxu-matrix and $F$ is a regular (non-singular) uxu-matrix (see the Appendix for more information on the generalized singular value decomposition).

By noting that

$$\|Ax\|^2 = \|US_A F^T x\|^2 = \|S_A F^T x\|^2$$
$$\|Bx\|^2 = \|VS_B F^T x\|^2 = \|S_B F^T x\|^2$$

and by defining $y = F^T x$, the original problem ($\|Ax\| = \min$ subject to $\|Bx\| = 1$) can be reduced to the equivalent problem

$$\|S_A y\| = \min \quad \text{subject to} \quad \|S_A y\| = 1$$

(15)

Assuming that the diagonal elements of $S_A$ are ordered as $\alpha_{i+1} \geq \alpha_i$ and the diagonal elements of $S_B$ are ordered as $\beta_{i+1} \leq \beta_i$, the solution of this problem is simply the vector

$$\hat{y} = [1/\beta_i \ 0 \ ... \ 0]^T$$

(16)

The LS-solution $\hat{x}$ can then be solved from the system

$$F^T \hat{x} = \hat{y}$$

(17)

Note that the LS-solution $\hat{x}$ is the first row of the inverse matrix $F^{-1}$ divided by the scalar $\beta_i$. Note also that the squared residual sum is simply

$$s_i^2 = \alpha_i^2 / \beta_i^2$$

(18)

i.e. the square of the first generalized singular value of $A$ and $B$.

We thus have the following algorithm for the LS-solution:

1. Compute the generalized singular value decomposition of $A$ and $B$ ($\rightarrow S_A$, $S_B$ and $F$)
2. Solve $\hat{x}$ from the system $F^T \hat{x} = \hat{y}$, where $\hat{y} = [1/\beta_i \ 0 \ ... \ 0]^T$.

Compared to the first algorithm, a favorable feature of this algorithm is that the computation of matrices $A^T A$ and $B^T B$ is avoided.
In the special case $B = I$ the solution simplifies again essentially. Let $A = USV^T$ be the singular value decomposition of $A$. Then $\|Ax\| = \min \iff \|USV^Tx\| = \min$ and furthermore $\|USV^Tx\| = \|SV^Tx\|$ and $\|x\| = \|V^Tx\|$. Thus, by defining $y = V^Tx$, the original problem reduces to the problem:

Find $\hat{y}$ such that $\|S\hat{y}\| = \min$ and $\|\hat{y}\| = 1$.

Because $S$ is a diagonal matrix and $s_i > s_{i+1}$, the LS-solution of this problem is simply $\hat{y} = [0 \ 0 \ \ldots \ 0 \ 1]^T$. Thus the LS-solution $\hat{x}$ of the original problem is

$$\hat{x} = V\hat{y} = V\begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = V_u$$

(19)

i.e. the last column $V_u$ of the matrix $V$ or the singular vector related to the smallest singular value of $A$.

4. AN EXAMPLE

We demonstrate the homogeneous LS-problem by fitting a straight line to given measured points $(x_i, y_i)$ using the model

$$ax + by + c = 0 \quad \text{or} \quad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

We consider the following constraints for the parameters:

- $b = 1$
- $\sqrt{a^2 + b^2 + c^2} = 1$
- $\sqrt{a^2 + b^2} = 1$

The first constraint leads to singular case, if the line is parallel to y-axis. The second constraint has no particular advantages apart from its simplicity. Third constraint on the contrary turns out very interesting (Hessian Normal Form). Note that the distance of a point $(x, y)$ from the line $ax + by + c = 0$ is $d = \left| (ax + by + c) / \sqrt{a^2 + b^2} \right|$. Thus, if $\sqrt{a^2 + b^2} = 1$, then $d = \left| ax + by + c \right|$. Therefore, when choosing this constraint, the residuals are orthogonal to the fitted line!
By defining \( x = \begin{bmatrix} a & b & c \end{bmatrix}^T \) we find easily that the constraint \( \sqrt{a^2 + b^2} = 1 \) is equivalent to 
\[ \|Bx\| = 1, \]
where
\[ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Given the points \{(1,2), (2,6), (6,1)\} we also have
\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 6 & 1 & 1 \end{bmatrix} \]

The computations using MATLAB give the LS-solution \( \hat{x} = \begin{bmatrix} 0.7071 \\ 0.7071 \\ -4.2426 \end{bmatrix} \). After appropriate scaling the fitted line can be written e.g. as
\[ y = -x + 6 \]

The residual vector is \( \hat{v} = A\hat{x} = \begin{bmatrix} -2.12 \\ 1.41 \\ 0.71 \end{bmatrix} \).

When compared to other methods for line fitting the following features of the presented methods are worth emphasizing:
- No singular cases (orientation of the line can be arbitrary).
- No need for approximate values and iterations.
- Orthogonal residuals are minimized.

5. REFERENCES


APPENDIX

Eigenvalue decomposition

Let $M$ be a symmetric xu-matrix. Then there exist an orthogonal matrix $E$ and a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$ ME = ED $$

Diagonal element $\lambda_i$ is the $i$th eigenvalue of $M$ and the column $e_i$ is the corresponding eigenvector.

Generalized eigenvalue decomposition

Let $M$ and $N$ be symmetric xu-matrices. Then there exists a regular xu-matrix $G$ and a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$ MG = NGD $$

Diagonal element $\lambda_i$ is $i$th generalized eigenvalue of $M$ and $N$. The column $g_i$ is the corresponding eigenvector.

Singular value decomposition

Let $A$ be a nxu-matrix ($n > u$). Then $A$ can be decomposed as

$$ A = USV^T \text{ or } A = U \begin{bmatrix} S' \\ 0 \end{bmatrix} V^T \text{ or } U^T AV = \begin{bmatrix} S' \\ 0 \end{bmatrix} $$

where $U$ is an orthogonal nxn-matrix, $V$ is an orthogonal nxu-matrix and $S'$ is a diagonal nxu-matrix.

Generalized singular value decomposition

Let $A$ be a nxu-matriisi and $B$ a pxu-matrix. Then $A$ and $B$ can be decomposed as (Mathworks, 2005)

$$ A = U S_A F_A^T \text{ or } B = V S_B F_B^T $$

where $U$ is an orthogonal nxn-matrix, $V$ is an orthogonal pxp-matrix, $F$ is a nxq-matrix ($q = \text{min}(n+p,u)$), $S_A$ is a “diagonal” nxu-matrix (diagonal elements $\alpha_{i+1} \geq \alpha_i$) and $S_B$ is a “diagonal” pxu-matrix (diagonal elements $\beta_{i+1} \leq \beta_i$).
Matrices $S_A$ and $S_B$ satisfy in this formulation the condition

$$S_A^T S_A + S_B^T S_B = 1$$

The structure of $S_A$ and $S_B$ depends on $n$, $p$, and $u$. For example, if $n > u$ and $p < u$, then

$$S_A = \begin{bmatrix} S_A' \\ 0 \end{bmatrix}, \quad S_A' = \text{diag}(\alpha_1, \ldots, \alpha_u), \quad \alpha_{i+1} \geq \alpha_i$$

and

$$S_B = \begin{bmatrix} S_B' \\ 0 \end{bmatrix}, \quad S_B' = \text{diag}(\beta_1, \ldots, \beta_p), \quad \beta_{i+1} \leq \beta_i$$

Note that, if $n > u$, then $F$ is a square matrix (uxu). In addition, if the rank of $[A^T \quad B^T]$ is full, then $F$ is a regular matrix and

$$U^T AF^{-T} = V^T BF^{-T} S$$

where

$$S = S_B^{-1} S_A = \text{diag}(\alpha_1/\beta_1, \ldots, \alpha_u/\beta_u)$$

Diagonal elements $s_i$ are the generalized singular values of $A$ and $B$. 